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NON-LINEAR LONG WAVES IN PROBLEMS WITH AXIAL SYMMETRY†

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Some classical types of waves on shallow water are investigated using the Boussinesq equation in polar coordinates. In these coordinate, normal perturbation theory methods lead to overdetermined systems of linear algebraic equations for unknown coefficients. It is shown that, the special cases examined, these equations are compatible, which makes it possible to construct solutions of Boussinesq equation with the same accuracy as that with which the equation was obtained. The velocity potential specified on the bottom and the function specifying the free surface of the water are expanded in a Fourier series in terms of time. The coefficients of their first two harmonics are expressed explicitly as polynomials in Bessel functions with coefficients in the form of elementary functions of the polar coordinates.

1. INTRODUCTION

Lamb [1, Sections 191–195] examines (in polar coordinates r, θ) at least three special cases of long, linear, three-dimensional waves: (1) axisymmetrical waves propagating over a horizontal bottom and caused by a periodic energy source [see below, formula (1.2)]; (2) the simplest non-axisymmetrical wave motion [formula (1.3)] in a circular tank; (3) rough model of 12-hour rising tides in a basin at the Earth's pole, bounded by a small circle of latitude [formula (1.4)].

Below, the solutions obtained for the Boussinesq equation are found with the same accuracy as that with which the equation was derived and can be regarded as non-linear corrections to the classical linear solutions mentioned.

On changing of the Boussinesq equation there is a reduction in the dimension of the problem, since the velocity potential is expanded in a power series in terms of the vertical coordinate. This expansion was used by Lagrange [2] and then developed by Bossinesq [3], and the modern form has been obtained by Friedrichs [4] (see the review [5]). Different versions of the Boussinesq equation are related mainly to the choice of the principal variables (see [6, 7]). Adopting Mei's notation [8], we use the velocity potential specified on the bottom $F(x, y, t)$, and the free surface elevation $\eta(x, y, t)$, as the principal variables. Note that the function η can be expressed in terms of F.

There are two small parameters related to the Boussinesq equation: ε – the ratio of the amplitude to the depth (a measure of the non-linearity), and μ – the ratio of the depth to the wavelength (the variance). As in the classical Boussinesq equation, we retain terms $O(\varepsilon)$ and $O(\mu^2)$ [but do not assume the equality $O(\varepsilon) = O(\mu^2)$].

The velocity potential specified on the bottom is expanded in a Fourier series in terms of the time

$$\begin{aligned} F(r, \theta, t) &= U(r, \theta) + S^1(r, \theta)\sin\omega t + C^1(r, \theta)\cos\omega t + S^2(r, \theta)\sin 2\omega t + \\ &+ C^2(r, \theta)\cos 2\omega t + \dots + S^m(r, \theta)\sin m\omega t + C^m(r, \theta)\cos m\omega t + \dots \end{aligned} \tag{1.1}$$

The main result of this paper consists of explicit expressions for the functions

$$U(r, \theta), \quad S^1(r, \theta), \quad C^1(r, \theta), \quad S^2(r, \theta), \quad C^2(r, \theta) \tag{1.2}$$

up to orders ε and μ^2 [see formulae (4.1), (4.2), (5.1), (5.2), (6.1) and (6.2)]. These functions are homogeneous polynomials in the Bessel function $Z_0(\omega r)$ and $Z_1(\omega r)$ and in trigonometric functions of the angular variable θ, and their coefficients are polynomials in r^{-1} and r. Similar formulae are also given for the function η. These expressions give periodic solutions of the Boussinesq equation with the same accuracy as that with which the equation was obtained. Therefore, the result can be interpreted as a

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periodic solution of the equations of surface waves [see below, Eqs (2.2–2.5)], found with accuracy $O(\varepsilon)$ and $O(\mu^2)$.

The three classical linear solutions [1] have the form

$$1)\ F(r, \theta, t) = J_0(\omega r)\sin\omega t + Y_0(\omega r)\cos\omega t \tag{1.3}$$

$$2)\ F(r, \theta, t) = J_1(\omega r)\cos\theta\sin\omega t \tag{1.4}$$

$$3)\ F(r, \theta, t) = J_2(\omega r)\cos 2\theta\sin\omega t \tag{1.5}$$

An attempt to find the functions (1.2) with accuracy $O(\varepsilon)$ and $O(\mu^2)$ necessitates solving a Bessel-type second-order inhomogeneous differential equation

$$Z''(r) + \frac{1}{r}Z'(r) + \left(A + \frac{B}{r^2}\right)Z = q_{00}J_0(\omega r)Y_0(\omega r) + q_{01}J_0(\omega r)Y_1(\omega r) + q_{10}J_1(\omega r)Y_0(\omega r) + q_{11}J_1(\omega r)Y_1(\omega r) \tag{1.6}$$

where q_{ij} represents polynomials in r and r^{-1}.

We will seek a solution in the form

$$Z(r) = Q_{00}J_0(\omega r)Y_0(\omega r) + Q_{01}J_0(\omega r)Y_1(\omega r) + Q_{10}J_1(\omega r)Y_0(\omega r) + Q_{11}J_1(\omega r)Y_1(\omega r) \tag{1.7}$$

where Q_{ij} represents polynomials in r and r^{-1} with unknown coefficients. These coefficients are calculated as solutions of an overdetermined system of linear algebraic equations (as in earlier papers [9, 10], where the same approach was used to describe, up to terms of order ε^2, $\varepsilon\mu^2$ and μ^4, long periodic waves over an inclined bottom). The general solution Eq. (1.6) is the sum of a particular solution, written out below, and the general solution of the corresponding homogeneous equation, which is a linear combination of Bessel functions.

In the cases examined, the compatibility of these overdetermined systems is a question of luck, and there is no obvious reason for the systems corresponding to higher harmonics also to be compatible. However, it can be assumed that the given expressions will be the first terms of the still unknown three-dimensional exact solution of Eqs (2.2)–(2.5). Exact three-dimensional solutions are unknown at present, but below an "intermediate" solution is proposed (a solution of order ε and μ^2) that, it is hoped, describes more accurately the behaviour of long periodic waves compared with Lamb's linear solution.

2. BASIC EQUATIONS

We will recall briefly the derivation of the Boussinesq equation in a form convenient for subsequent presentation and introduce notation following, principally, Mei [8]. The dimensionless quantities are introduced in the following way

$$x = \frac{x'}{l'_0},\quad y = \frac{y'}{l'_0},\quad z = \frac{z'}{h'_0},\quad t = \frac{g^{\frac{1}{2}}h_0'^{\frac{1}{2}}}{l'_0}t',\quad \eta = \frac{\eta'}{a'_0},\quad \varphi = \frac{h'_0}{a'_0 l'_0 g^{\frac{1}{2}} h_0'^{\frac{1}{2}}}\varphi',\quad h = \frac{h'}{h'_0} \tag{2.1}$$

The primes here correspond to physical variables: a'_0 is the characteristic amplitude of the wave, h'_0 is the depth, l'_0 is the wavelength, g is the acceleration due to gravity, x' and y' are the horizontal coordinates, z' is the vertical coordinate at t' is the time. The scaled equation and the boundary conditions for irrotational wave motion have the form

$$\varphi_{xx} + \varphi_{yy} + \mu^{-2}\varphi_{zz} = 0, \quad -1 < z < \varepsilon\eta(x, y, t) \tag{2.2}$$

$$\eta_t + \varepsilon\varphi_x\eta_x + \varepsilon\varphi_y\eta_y - \mu^{-2}\varphi_z = 0, \quad z = \varepsilon\eta(x, y, t) \tag{2.3}$$

$$\varphi_t + \eta + \frac{1}{2}\varepsilon(\varphi_x^2 + \varphi_y^2) + \varepsilon\mu^{-2}\varphi_x^2 = 0, \quad z = \varepsilon\eta(x, y, t) \tag{2.4}$$

$$\varphi_z = 0, \quad z = -1 \tag{2.5}$$

where ε and μ are measures of the non-linearity and variance, defined by the formulae

$$\varepsilon = a_0'/h_0', \quad \mu = h_0'/l_0' \tag{2.6}$$

The potential $\varphi(x, y, z, t)$ is expanded in powers of the vertical coordinate

$$\varphi(x, y, z, t) = \sum_{m=0}^{\infty} (z+1)^m F_m(x, y, t) \tag{2.7}$$

Substituting expression (2.7) into Eq. (2.2) and equation each power of $z + 1$ to zero, we obtain [the symbol ∇ is used to denote the horizontal gradient $(\partial/\partial x, \partial/\partial y)$]

$$F_{m+2} = -\frac{\mu^2\nabla^2 F_m}{(m+2)(m+1)}, \quad m = 0, 1, 2, \ldots \tag{2.8}$$

The boundary conditions on the bottom (2.5) give

$$F_1 = 0 \tag{2.9}$$

Thus, φ can be expressed in terms of $F_0(x, y, t)$

$$\varphi = F - \frac{1}{2!}\mu^2(z+1)^2\nabla^2 F + \frac{1}{4!}\mu^4(z+1)^4\nabla^4 F + O(\mu^6) \tag{2.10}$$

the (zero subscript in the function $F_0(x, y, t)$ has been omitted here and below).

Expression (2.10) satisfies Eq. (2.2) and boundary condition (2.5). Substituting expression (2.10) into conditions (2.3) and (2.4) we obtain the Boussinesq equations for the two functions: the potential on the bottom $F(x, y, t)$ and the free surface elevation $\eta(x, y, t)$

$$\eta_t + \varepsilon\nabla\eta \bullet \nabla F + (1 + \varepsilon\eta)\nabla^2 F - \frac{1}{6}\mu^2\nabla^2\nabla^2 F = 0 \tag{2.11}$$

$$\eta + F_t - \frac{1}{2}\mu^2\nabla^2 F_t + \frac{1}{2}\varepsilon(\nabla F)^2 = 0 \tag{2.12}$$

These equations are equivalent to the equations given by Mei [8], Ch. 11, Eqs (1.16) and (1.17).

In order to express the function specifying the free surface elevation $\eta(x, y, t)$ in terms of the functions $F(x, y, t)$ and its derivatives, we use the expansion

$$\eta = \eta_0 + \eta_2\mu^2 + \eta_4\mu^4 + O(\mu^6)$$

The substituting this expression into Eq. (2.12) we obtain the formulae

$$\eta_0 = -F_t - \frac{1}{2}(F_x^2 + F_y^2)\varepsilon, \quad \eta_2 = \frac{1}{2}(F_{xxt} + F_{yyt}) \tag{2.13}$$

Subsequent substitution of expressions (2.13) into Eq. (2.11) yields a unique equation for the function F

$$-F_{tt} + F_{xx} + F_{yy} + \left(\frac{1}{2}F_{xxtt} + \frac{1}{2}F_{yytt} - \frac{1}{6}F_{xxxx} - \frac{1}{6}F_{yyyy} - \frac{1}{3}F_{xxyy}\right)\mu^2 +$$
$$+ (-2F_xF_{xt} - 2F_yF_{yt} - F_{xx}F_t - F_{yy}F_t)\varepsilon = 0 \tag{2.14}$$

or in polar coordinates (r, θ)

$$-F_{tt}+\frac{1}{r^2}F_{\theta\theta}+\frac{1}{r}F_r+F_{rr}+\left(-\frac{2}{r^2}F_\theta F_{\theta t}-\frac{1}{r^2}F_tF_{\theta\theta}-\frac{1}{r}F_tF_r-2F_rF_{rt}-F_tF_{rr}\right)\varepsilon+$$
$$+\left[-\frac{2}{3r^4}F_{\theta\theta}+\frac{1}{2r^2}F_{\theta\theta tt}-\frac{1}{6r^4}F_{\theta\theta\theta\theta}-\frac{1}{6r^3}F_r+\frac{1}{2r}F_{rtt}+\frac{1}{3r^3}F_{r\theta\theta}+\frac{1}{6r^2}F_{rr}+\frac{1}{2}F_{rrtt}-\right.$$
$$\left.-\frac{1}{3r^2}F_{rr\theta\theta}-\frac{1}{3r}F_{rrr}-\frac{1}{6}F_{rrrr}\right]\mu^2=0 \tag{2.15}$$

3. THE PERIODIC PROBLEM

We will assume that the solution of Eq. (2.15) is periodic with respect to time and can be expanded in a Fourier series in a certain region, excluding the neighbourhood of the axis of symmetry, i.e.

$$F(r,\theta,t)=u(r,\theta)\varepsilon+(S_{00}^1(r,\theta)+S_{02}^1(r,\theta)\mu^2)\sin\omega t+(C_{00}^1(r,\theta)+C_{02}^1(r,\theta)\mu^2)\cos\omega t+$$
$$+(S_{10}^2(r,\theta)\varepsilon)\sin 2\omega t+(C_{10}^2(r,\theta)\varepsilon)\cos 2\omega t+\ldots \tag{3.1}$$

The form of the coefficients of $\sin m\omega t$ and $\cos m\omega t$ is determined by recurrent calculations using the solution of Eq. (2.15).

In the zeroth order we have the following linear problems for $S_{00}^1(r,\theta)$ and $C_{00}^1(r,\theta)$

$$\omega^2S_{00}^1+\frac{1}{r^2}S_{00\theta\theta}^1+\frac{1}{r}S_{00r}^1+S_{00rr}^1=0 \tag{3.2}$$

$$\omega^2C_{00}^1+\frac{1}{r^2}C_{00\theta\theta}^1+\frac{1}{r}C_{00r}^1+C_{00rr}^1=0 \tag{3.3}$$

Their solutions, expressed in polar coordinates, can be represented in the form of the series

$$\begin{aligned}&\alpha_0J_0(\omega r)+\beta_0Y_0(\omega r)+(\alpha_1J_1(\omega r)+\beta_1Y_1(\omega r))\cos\theta+\\&+\ldots+(\alpha_nJ_n(\omega r)+\beta_nY_n(\omega r))\cos n\theta+\ldots\end{aligned} \tag{3.4}$$

We will concentrate on three cases, corresponding to formulae (1.3)–(1.5).

Case 1

$$S_{00}^1=\alpha_0J_0(\omega r)+\beta_0Y_0(\omega r),\quad C_{00}^1=\gamma_0J_0(\omega r)+\delta_0Y_0(\omega r) \tag{3.5}$$

Case 2

$$S_{00}^1=\alpha_1J_1(\omega r)\cos\theta,\quad C_{00}^1=0 \tag{3.6}$$

Case 3

$$S_{00}^1=\alpha_2J_2(2\omega r)\cos 2\theta,\quad C_{00}^1=0 \tag{3.7}$$

The first solution was used to describe axisymmetrical wave motion with a periodic source at the centre of the system of polar coordinates [1, Sections 191–195]. The second solution was used to describe the simplest (but "most interesting") case of regular non-axisymmetrical wave motion in a circular tank. The third solution yields a crude presentation of 12-hour rising tides for a basin at the Earth's pole, bounded by a small circle of latitude. The aim of our subsequent examination is to provide the next-order correction to these classical solutions.

We will denote by $S=S(r)$ and $C=C(r)$ the two solutions of Bessel's equation

$$rZ_{rr}+Z_r+\omega^2rZ=0 \tag{3.8}$$

and their derivatives will be denoted by S'' and C''. The functions $S(r)$, $C(r)$, $S'(r)$ $C'(r)$ can be represented in terms of Bessel functions as follows:

$$S(r)=a_{11}J_0(\omega r)+a_{12}Y_0(\omega r),\quad S'(r)=\omega(-a_{11}J_1(\omega r)-a_{12}Y_1(\omega r))$$
$$C(r)=a_{21}J_0(\omega r)+a_{22}Y_0(\omega r),\quad C'(r)=\omega(-a_{21}J_1(\omega r)-a_{22}Y_1(\omega r))$$

4. CASE 1: AXISYMMETRICAL WAVES WITH A PERIODIC SOURCE

In the axisymmetrical case, formulae (1.1) acquires the form

$$F(r,t) = u(r)\varepsilon + (S_{00}^1(r) + S_{02}^1(r)\mu^2)\sin\omega t + (C_{00}^1(r) + C_{02}^1(r)\mu^2)\cos\omega t + \\ + S_{10}^2(r)\varepsilon\sin 2\omega t + C_{10}^2(r)\varepsilon\cos 2\omega t + \ldots, \quad S_{00}^1(r) = S, \quad C_{00}^1 = C \tag{4.1}$$

Calculations yield explicit formulae for the coefficients

$$u = 0$$
$$S_{02}^1 = \frac{\omega^2 r}{6}S', \quad C_{02}^1 = \frac{\omega^2 r}{6}C' \tag{4.2}$$
$$S_{01}^2 = \frac{\omega}{2}(S^2 - C^2) + \frac{3\omega}{4}r(SS' - CC'), \quad C_{10}^2 = \omega CS + \frac{3\omega}{4}r(SC' + S'C)$$

Substituting these expressions into formulae (3.12), we obtain

$$\eta(r,\theta,t) = L_{10}^0(r)\varepsilon + (P_{00}^1(r) + P_{02}^2(r)\mu^2)\sin\omega t + (Q_{00}^1(r) + Q_{02}^1(r)\mu^2)\cos\omega t + \\ + (P_{10}^2(r)\varepsilon)\sin 2\omega t + (Q_{10}^2(r)\varepsilon)\cos 2\omega t \tag{4.3}$$

Here

$$L_{10}^0 = -\frac{1}{4}S'^2 - \frac{1}{4}C'^2$$
$$P_{00}^1 = \omega C, \quad P_{02}^1 = \frac{\omega^3}{2}C + \frac{\omega^3 r}{6}C', \quad Q_{00}^1 = -\omega S, \quad Q_{02}^1 = -\frac{\omega^3}{2}S - \frac{\omega^3 r}{6}S' \tag{4.4}$$
$$P_{10}^2 = 2\omega^2 CS + \frac{3\omega^2 r}{2}(CS' + C'S)$$
$$Q_{10}^2 = \omega^2(C^2 - S^2) + \frac{3\omega^2 r}{2}(CC' - SS') - \frac{1}{4}(C'^2 - S'^2)$$

Special cases
Assuming

$$a_{12} = -\omega^{-1}, \quad a_{21} = \omega^{-1}, \quad a_{11} = a_{22} = 0 \tag{4.5}$$

we have $S(r) = -\omega^{-1}Y_0(\omega r)$ and $C(r) = \omega^{-1}J_0(\omega r)$. Then, in the principal approximation, we obtain

$$\eta(r,\theta,t) = J_0(\omega r)\sin\omega t + Y_0(\omega r)\cos\omega t$$

Consequently, when $r \to +\infty$

$$\eta(r,\theta,t) \equiv \sqrt{\frac{2}{\pi\omega r}}\left(\cos\left(\omega r - \frac{\pi}{4}\right)\sin\omega t + \sin\left(\omega r - \frac{\pi}{4}\right)\cos\omega t\right) = \sqrt{\frac{1}{2\pi\omega r}}\sin\left(\omega r - \frac{\pi}{4} + \omega t\right)$$

Thus, case (4.5) corresponds to a progressive wave while the case

$$a_{11} = 1, \quad a_{22} = a_{12} = a_{21} = 0 \tag{4.6}$$

corresponds to a standing (axisymmetrical) wave.

An axisymmetrical progressive wave generated by a point source in shown in Fig. 1 for $\omega = 0.6$, $\varepsilon = 0.1$, $\mu = 0.3$ and $t = \pi/\omega$. The graph illustrates the dependence of the free surface elevation η on the radial coordinate r. The continuous curve is the non-linear solution (of order ε and μ^2), and the dashed curve is the linear solution. On the wave corresponding to the new solution, the front slope is steeper, while the rear slope is shallower.

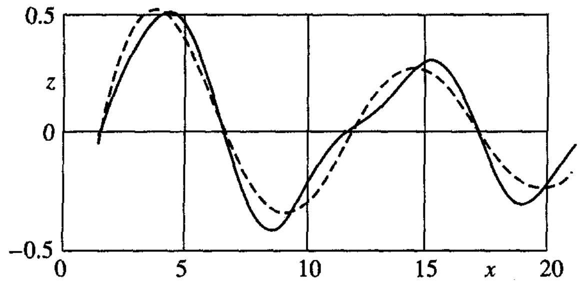

Fig. 1

5. CASE 2: NON-AXISYMMETRICAL WAVES (STANDING WAVES IN A CIRCULAR TANK)

We will assume that the solution is periodic with respect to time and can be expanded in a Fourier series in the region obtained by eliminating the neighbourhood of the axis of symmetry:

$$\begin{aligned} F(r,\theta,t) &= u(r,\theta)\varepsilon + (C_{00}^{1}(r,\theta) + C_{02}^{1}(r,\theta)\mu^2)\cos\omega t + (S_{10}^{2}(r,\theta)\varepsilon)\sin 2\omega t + \dots, \\ C_{00}^{1} &= C'\cos\theta \end{aligned} \tag{5.1}$$

Calculations yield

$$\begin{aligned} u &= 0 \\ C_{02}^{1} &= -\frac{\omega^2}{6}(r\omega^2 C + C')\cos\theta \\ S_{10}^{2} &= \left(\frac{3\omega^3}{8}C^2 + \frac{3\omega^3}{8}rCC' - \frac{\omega}{4}C'^2\right) + \left(\frac{3\omega^3}{8}rCC' + \frac{\omega}{8}C'^2\right)\cos 2\theta \end{aligned} \tag{5.2}$$

The function $C = C(r)$ is again the solution of Eq. (3.8).

Substituting these expressions into formulae (2.13), we obtain

$$\eta(r,\theta,t) = L_{10}^{0}(r,\theta)\varepsilon + (P_{00}^{1}(r,\theta) + P_{02}^{1}(r,\theta)\mu^2)\sin\omega t + (Q_{10}^{2}(r,\theta)\varepsilon)\cos 2\omega t \tag{5.3}$$

Here

$$\begin{aligned} L_{10}^{0} &= -\frac{\omega^4}{4}C^2\cos^2\theta - \frac{\omega^2}{2r}CC'\cos^2\theta - \frac{1}{4r^2}C'^2 \\ P_{00}^{1} &= \omega C'\cos\theta, \quad P_{02}^{1} = -\frac{\omega^5}{6}rC\cos\theta + \frac{\omega^3}{3}C'\cos\theta \\ Q_{10}^{2} &= \omega^4 C^2\left(-\frac{1}{4}\cos^2\theta - \frac{3}{4}\right) - \omega^2 CC'\left(\frac{1}{2r}\cos^2\theta + \frac{3\omega^2}{2}r\cos^2\theta\right) + \\ &+ C'^2\left(-\frac{1}{4r^2} + \frac{3}{4}\omega^2 - \frac{1}{2}\omega^2\cos^2\theta\right) \end{aligned} \tag{5.4}$$

The non-linear corrections proposed considerably alter the shape of the level lines. In particular, the surface is never plane, unlike the classical linear solution.

The solution obtained for $\omega = 1.0$, $\varepsilon = 0.2$ and $\mu = 0.4$ is illustrated in Fig. 2, where the contours (level lines $\eta = 0.1, 0.2, 0.3, 0.4$) of non-axisymmetrical waves (Case 2) for different values of t are presented. The continuous curves are the solutions of order ε and μ^2, and the dashed curves are the classical Lamb solution.

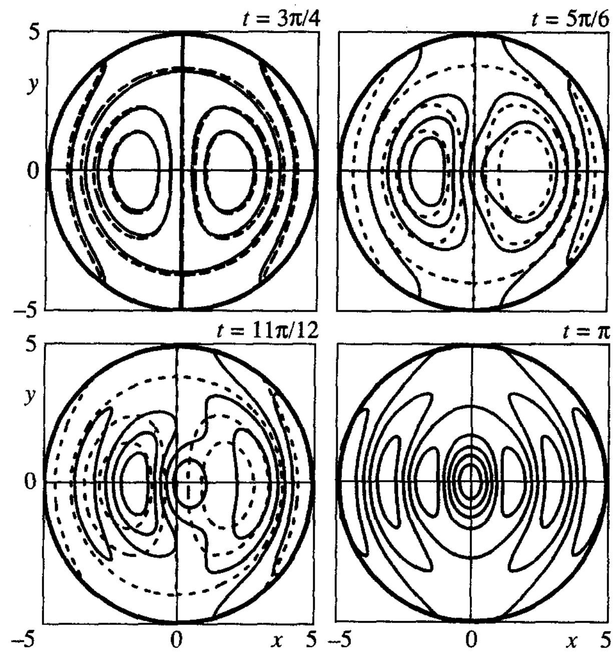

Fig. 2

6. CASE 3: NON-AXISYMMETRICAL WAVES (12-HOUR RISING TIDES)

Case 3 differs from Case 2 in that

$$C_{00}^{1} = \left(-\omega^2 C - \frac{2}{r}C'\right)\cos 2\theta \tag{6.1}$$

The following formulae are obtained

$$u = 0$$

$$C_{02}^{1} = -\frac{\omega^4}{6} r C' \cos 2\theta$$

$$S_{10}^{2} = \omega^4 C^2 + \left(\frac{3\omega^4 r}{8} + \frac{\omega^2}{r}\right)CC' + \left(-\frac{3\omega^2}{4} + \frac{1}{r^2}\right)C'^2 + \tag{6.2}$$

$$+\left[-\frac{\omega^4}{2}C^2 + \left(-\frac{2\omega^2}{r} + \frac{3\omega^4 r}{8}\right)CC' + \left(-\frac{2}{r^2} + \frac{3\omega^2}{4}\right)C'^2\right]\cos 4\theta$$

Substituting these expressions into formulae (2.13), we obtain

$$\eta(r, \theta, t) = L_{10}^{0}(r, \theta)\varepsilon + (P_{00}^{1}(r, \theta) + P_{02}^{1}(r, \theta)\mu^2)\sin\omega t + Q_{10}^{2}(r, \theta)\varepsilon\cos 2\omega t \tag{6.3}$$

Here

$$L_{10}^{0} = \left(-\frac{\omega^4}{2} - \frac{\omega^4}{2r^2}\right)C^2 + \left(-\frac{2\omega^2}{r^3} - \frac{2\omega^2}{r} + \frac{\omega^4}{2r}\right)CC' + \left(-\frac{2}{r^4} - \frac{2}{r^2} + \frac{\omega^2}{r^2} - \frac{\omega^4}{8}\right)C'^2 +$$

$$+\left[\left(\frac{\omega^4}{2} - \frac{\omega^4}{2r^2}\right)C^2 + \left(-\frac{2\omega^2}{r^3} + \frac{2\omega^2}{r} + \frac{\omega^4}{2r}\right)CC' + \left(-\frac{2}{r^4} + \frac{2}{r^2} + \frac{\omega^2}{r^2} - \frac{\omega^4}{8}\right)C'^2\right]\cos 4\theta$$

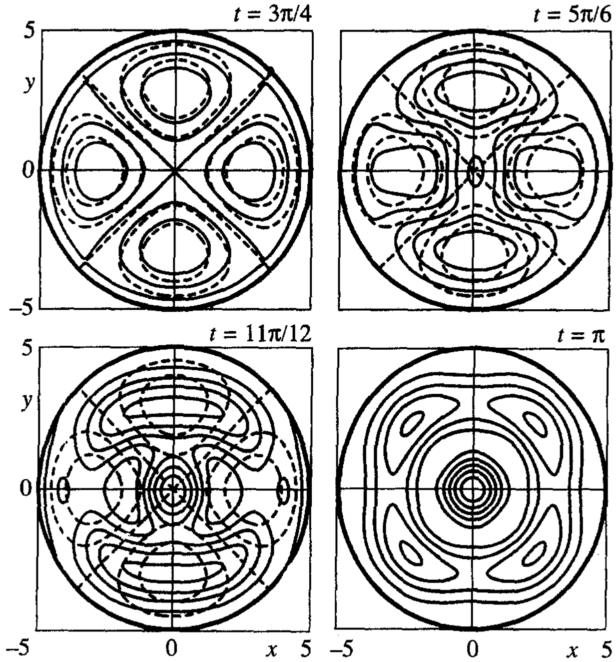

Fig. 3

$$P^1_{00} = \omega\left(-\omega^2 C - \frac{2}{r}C'\right)\cos 2\theta, \quad P^1_{02} = \omega\left(-\frac{\omega^4}{2}C - \left(\frac{\omega^2}{r} + \frac{\omega^4 r}{6}\right)C'\right)\cos 2\theta$$

$$Q^2_{10} = \left(-\frac{\omega^4}{2} - \frac{\omega^4}{2r^2} - 2\omega^5\right)C^2 + \left(-\frac{2\omega^2}{r^3} - \frac{2\omega^2}{r} - \frac{2\omega^3}{r} + \frac{\omega^4}{2r} - \frac{3r\omega^5}{4}\right)CC' +$$

$$+\left(-\frac{2}{r^4} + \frac{2}{r^2} + \frac{4\omega}{r^2} + \frac{\omega^2}{r^2} - \frac{3\omega^3}{2} - \frac{\omega^4}{8}\right)C'^2 +$$

$$+\left[\left(\frac{\omega^4}{2} - \frac{\omega^4}{2r^2} + \omega^5\right)C^2 + \left(-\frac{2\omega^2}{r^3} - \frac{2\omega^2}{r} - \frac{2\omega^3}{r} + \frac{\omega^4}{2r} - \frac{3r\omega^5}{4}\right)CC' +\right.$$

$$\left.+\left(-\frac{2}{r^4} - \frac{2}{r^2} - \frac{2\omega}{r^2} + \frac{\omega^2}{r^2} + \frac{3\omega^3}{2} - \frac{\omega^4}{8}\right)C'^2\right]\cos 4\theta$$

The solutions obtained for $\omega = 1.0$, $\varepsilon = 0.2$ and $\mu = 0.4$ is illustrated in Fig. 3, where the contours (level lines $\eta = 0.1, 0.2, 0.3, 0.4$) of non-axisymmetrical waves (Case 3) for different values of t are presented. The continuous curves are the solution of order ε and μ^2, and the dashed curves are the classical linear Lamb solution.

7. CONCLUSION

There are at least two methods for reducing the three-dimensional problem of surface waves to a two-dimensional problem:

(1) assume that the motion is not dependent on one of the horizontal coordinates;

(2) assume that the depth is small compared with the wavelength; this enables one to eliminate the vertical coordinate from the number of independent variables in approximate shallow water theories (Lagrangian approximations). Both these methods eliminate from consideration one of the spatial coordinates.

The second method, leading to classical Boussinesq-type equations, has been used above. For these equations, three types of periodic solution have been obtained. They can be regarded as solutions of the classical equations of surface waves (2.2)–(2.5) up to terms of order ε and μ^2. Intermediate equations are given for illustrating the method of derivation, but solutions corresponding to Cases 1 to 3 can be checked by substituting into system (2.2)–(2.5) (using expression (2.10) for the potential). Linear versions of these problems were the subject of the classical investigation in Lamb's book [1, Sections 191–195].

We assume that these expressions are only the first terms of a certain series giving an accurate three-dimensional solution of the equations of surface waves (2.2)–(2.5). The number of known accurate solutions is fairly small; in particular, no three-dimensional solutions are known.

The results were obtained by the method of undetermined coefficients as solutions of overdetermined systems of algebraic linear equations (the cause of their compatibility remains unclear). These results can be interpreted as the integrability of certain cubic expressions of Bessel functions.

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